# WEAK COMPACTNESS AND REFLEXIVITY[1]

BY

ROBERT C. JAMES

ABSTRACT

Several characterizations of weak-compactness are given for subsets of complete locally convex linear topological spaces and of Banach spaces. Some are new and some are generalizations of known facts.

**Introduction.** The purpose of this paper is to study characterizations of weak compactness of bounded sets in complete real locally convex linear topological spaces and in real Banach spaces and to use these characterizations of weak compactness to obtain characterizations of reflexive Banach spaces. Some of the characterizations are new, many are improved or established for more general cases than previously, and some are known but are proved here both as useful steps in sequences of implications and to demonstrate relations to other characterizations. Only properties equivalent to w-compactness will be discussed. For bounded w-closed subsets $E$ of a locally convex linear topological space, each of (1)–(17) is equivalent to w-compactness. If $E$ also is convex, we can include (18)–(25) as well, and also (26) if $E$ contains 0. For bounded w-closed subsets of a Banach space, each of (1)–(17) and (27)–(28) is equivalent to w-compactness. Each of (1)–(38) is a necessary and sufficient condition for reflexivity of a Banach space, with $E$ understood to be the unit ball.

The symbols $X + Y$ and $X - Y$ will denote the sets of all sums $x + y$ and all differences $x - y$ with $x$ in $X$ and $y$ in $Y$. The abbreviations "w-closed" and "w-compact" will be used for weakly closed and weakly compact (i.e., closed and compact relative to the weak topology). Also, "wc-compact" will be used for weakly countably compact and "ws-compact" for weakly sequentially compact. Since all spaces used are $T_1$ (and in fact Hausdorff), countably compact can mean either that any countable open cover contains a finite cover or that each infinite subset has an accumulation point in the set. A weakly sequentially compact set is a set such that each sequence in the set contains a subsequence that converges weakly to a point in the set. The closure of a set $E$ will be denoted by cl($E$) and the linear span of a set $E$ by lin($E$). The convex span of $E$ will be

Received August 19, 1964.

(1) Lecture delivered at a symposium on Series and Geometry in Linear Spaces, held at the Hebrew University of Jerusalem from March 16, till March 24, 1964.

denoted by conv($E$) and is the set of all finite sums $\sum\alpha_i x_i$ with each $\alpha_i$ nonnegative, $\sum\alpha_i = 1$, and each $x_i$ in $E$. The circled span of $E$ will be denoted by cir($E$) and is the set of all finite sums $\sum a_i x_i$ with $\sum|a_i| \leqq 1$ and each $x_i$ in $E$. The flat span of $E$ will be denoted by flat($E$) and is the set of all finite sums $\sum a_i x_i$ with $\sum a_i = 1$ and each $x_i$ in $E$. Several applications will be made of the following elementary facts.

THEOREM ON STRONG SEPARATION [12, Theorem 14.3, p. 118]. *If $T$ is a locally convex linear topological space and $X$ and $Y$ are nonempty disjoint convex subsets of $T$, then 0 is not a member of* cl$[Y - X]$ *iff there is a continuous linear functional $f$ such that*

$$\sup\{f(x): x \in X\} < \inf\{f(y): y \in Y\}.$$

A useful special case of this theorem states that if $K$ is a closed convex subset of $T$, then for any member $x$ of $T$ not in $K$ there is a continuous linear functional $f$ with $f(x) > \sup\{f(y): y \in K\}$ [2, Theorem 5, p. 22].

HELLY'S CONDITION (see [2, Theorem 3, p. 38], and [12, pp. 151–2]). *If $T$ is a normed linear space and $f_1, \cdots, f_n$ are linear functionals on $T$, then for numbers $c_1, \cdots, c_n$ and $\varepsilon > 0$ there exists an $x$ in $T$ with $\|x\| < M + \varepsilon$ and $f_k(x) = c_k$ for each $k$ iff*

$$\left| \sum_1^n a_i c_i \right| < M \left\| \sum_1^n a_i f_i \right\|$$

*for all numbers $a_1, \cdots, a_n$.*

LEMMA. *For any bounded sequence $\{x_n\}$ in a normed linear space $T$, there is an $F$ in $T^{**}$ such that*

$$\underline{\lim}\ f(x_n) \leqq F(f) \leqq \overline{\lim}\ f(x_n) \text{ for all } f \text{ in } T^*.$$

**Proof.** Let $\mu$ be a linear functional of unit norm on the space $m(\omega)$ of bounded sequences with the property that

$$\underline{\lim}\ t_n \leqq \mu(t) \leqq \overline{\lim}\ t_n$$

for all $t = \{t_1, \cdots\}$. Then we can define $F$ that satisfies the desired inequality by letting $F(f) = \mu(\{f(x_1), \cdots\})$ for all $f$ in $T^*$. The functional $\mu$ can be any linear functional of unit norm on $m(\omega)$ with $\mu(t) = \lim t_n$ whenever this limit exists, or $\mu$ can be a Banach limit [2, Theorem 2, p. 83].

1. **Bounded w-closed subsets of complete locally convex linear topological spaces.** Most of the conditions listed as (1)–(9) of Theorem 1 have been studied previously, but some only for less general cases. It has long been known that

a Banach space is reflexive iff (3) is satisfied for $E$ the unit sphere and $\{x_n\}$ an arbitrary transfinite sequence in $E$ [5], and iff (4) is satisfied for $E$ the unit sphere (see [15] and [20]). However, the equivalence of (1) with each of (2), (3) and (4) is now well known (see [2, Corollary 2, p. 51], and [12, Theorem 17.12, p. 159]). Condition (6) is closely related to the well known condition (5) (see [6, pp. 177 and 185], and [12, Theorem 17.12, p. 159]). Conditions of type (7) and (8) were developed first for the unit sphere of a Banach space [8, Lemma 1, p. 159] and later for bounded closed convex subsets of a Banach space [11, Theorems 1 and 2, pp. 130–131]. Condition (9) was developed first for the unit sphere of a Banach space [9, Theorem 1, p. 206] and is closely related to the later concept of a cone isomorphic to the cone spanned by unit members of $l^1$ discussed in [16] as a condition for nonreflexivity of Banach spaces. Condition (9) has also been studied for bounded closed convex subsets of a Banach space [11, Theorem 3, p. 132].

THEOREM 1. *Let $T$ be a complete locally convex linear topological space and let $E$ be a bounded* w*-closed subset of $T$. Then the following are equivalent:*

(1) *$E$ is* w*-compact.*

(2) *$E$ is* wc*-compact.*

(3) *For each sequence $\{x_n\}$ in $E$ there is an $x$ in $E$ such that, for all continuous linear functionals $f$,*

$$\underline{\lim}\, f(x_n) \leqq f(x) \leqq \overline{\lim}\, f(x_n).$$

(4) *If $\{K_n\}$ is a nested sequence of closed convex sets and $E \cap K_n$ is nonempty for each $n$, then $E \cap (\cap K_n)$ is nonempty.*

(5) *For each sequence $\{x_n\}$ in $E$ and each equicontinuous sequence $\{f_n\}$ of linear functionals,*

$$\lim_n \lim_k f_n(x_k) = \lim_k \lim_n f_n(x_k)$$

*whenever all of these limits exist.*

(6) *For each sequence $\{x_n\}$ in $E$ and each equicontinuous sequence $\{f_n\}$ of linear functionals,*

$$\inf\{f_n(x_k) \colon n < k\} \leqq \sup\{f_n(x_k) \colon n > k\}.$$

(7) *For each sequence $\{x_n\}$ in $E$, the member $0$ of $T$ belongs to*

$$\mathrm{cl}\left[\bigcup_{n=1}^{\infty} (\mathrm{conv}\{x_1, \cdots, x_n\} - \mathrm{conv}\{x_{n+1}, \cdots\})\right].$$

(8) *For each sequence $\{x_n\}$ in $E$, the member $0$ of $T$ belongs to*

$$\mathrm{cl}\left[\bigcup_{n=1}^{\infty} (\mathrm{lin}\{x_1, \cdots, x_n\} - \mathrm{conv}\{x_{n+1}, \cdots\})\right].$$

(9) *There does not exist a positive number $\theta$, a sequence $\{z_n\}$ in $E$, and an equicontinuous sequence $\{g_n\}$ of linear functionals such that*

$$g_n(z_k) > \theta \text{ if } n \leqq k, \qquad g_n(z_k) = 0 \text{ if } n > k.$$

**Proof.** Clearly (1) $\Rightarrow$ (2), since any $T_1$-space is countably compact if it is compact; (2) $\Rightarrow$ (3) follows from the fact if $x$ is an accumulation point of $\{x_n\}$, then the inequality in (3) is satisfied for all $f$.

To show that (3) $\Rightarrow$ (4), let $\{K_n\}$ be a nested sequence of closed convex sets and $E \cap K_n$ be nonempty for each $n$. Choose $x_n$ from $E \cap K_n$ for each $n$. If $x$ in $E$ satisfies the inequality in (3) for all $f$, then $x \in \bigcap K_n$. For if there is an $n$ such that $x \notin K_n$, then there is a continuous linear functional $\Phi$ with $\Phi(x) > \sup\{\Phi(y): y \in K_n\}$. This contradicts (3), since then $\overline{\lim}\, \Phi(x_n) < \Phi(x)$.

We shall complete the proof of Theorem 1 by showing that:

$$(4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (9), \qquad (4) \Rightarrow (7) \Rightarrow (8) \Rightarrow (9), \qquad (9) \Rightarrow (1).$$

To show that (4) $\Rightarrow$ (5), let $\{x_n\}$ be a sequence in $E$, let $\{f_n\}$ be an equicontinuous sequence of linear functionals for which all the limits in (5) exist, and use (4) to obtain an $x$ that belongs to $\mathrm{cl}[\mathrm{conv}\{x_{n+1}, \cdots\}]$ for all $n$. Then $\lim_k f_n(x_k) = f_n(x)$ for all $n$, and therefore

$$\lim_n \lim_k f_n(x_k) = \lim_n f_n(x).$$

If $L = \lim_k \lim_n f_n(x_k)$ and for a positive number $\varepsilon$ we choose $K$ so that

$$\left| L - \lim_n f_n(x_k) \right| < \varepsilon \qquad \text{if } k > K,$$

then $\left| L - \lim_n f_n(x) \right| \leqq \varepsilon$. Therefore $L = \lim_n f_n(x)$. The implication (5) $\Rightarrow$ (6) is easy. For if $\{x_n\}$ is a sequence in $E$ and $\{f_n\}$ is an equicontinuous sequence of linear functionals, then $\left| f_n(x_k) \right|$ is bounded and there is a subsequence $\{(x_n', f_n')\}$ of $\{(x_n, f_n)\}$ for which all the limits in (5) exist. Then the equality in (5) for $\{x_n'\}$ and $\{f_n'\}$ implies the inequality in (6). To show that (6) $\Rightarrow$ (9), we suppose (9) is false and let $\theta$, $\{z_n\}$, and $\{g_n\}$ be as described in (9). Then

$$\inf\{g_n(z_k): n < k\} \geqq \theta > 0 = \sup\{g_n(z_k): n > k\},$$

which contradicts (6).

The implication (4) $\Rightarrow$ (7) has a direct proof: If $\{x_n\}$ is a sequence in $T$, then using (4) there is an $x$ that is a member of $\mathrm{cl}[\mathrm{conv}\{x_{n+1}, \cdots\}]$ for all $n$. Then for an arbitrary neighborhood $W$ of 0, we choose a neighborhood $U$ of 0 such that $U - U \subset W$. For some $n$, there is an $r$ in $\mathrm{conv}\{x_1, \cdots, x_n\}$ with $x - r$ in $U$. For this (and any other) $n$, there is an $s$ in $\mathrm{conv}\{x_{n+1}, \cdots\}$ with $x - s$ in $U$. Then $W$ contains $r - s$ and (7) is verified. The implication (7) $\Rightarrow$ (8) is formal. To show that (8) $\Rightarrow$ (9), we suppose (9) is false and let $\theta$, $\{z_n\}$, and $\{g_n\}$ be as described in (9). Let $W$ be a neighborhood of 0 chosen so that $|g_n(x)| < \theta$ for all $n$ if $x \in W$. Then for any numbers $\{a_i\}$ and any nonnegative numbers $\{\alpha_i\}$ with $\sum_{n+1} \alpha_i = 1$, we have

$$\left| g_{n+1}\left( \sum_1^n a_i z_i - \sum_{n+1} \alpha_i z_i \right) \right| = \sum_{n+1} \alpha_i g_{n+1}(z_i) > \theta.$$

Therefore $W \bigcap [\bigcup_{n=1}^{\infty} (\operatorname{lin}\{x_1, \cdots, x_n\} - \operatorname{conv}\{x_{n+1}, \cdots\})]$ is empty.

It remains to show that (9) implies (1). First we use the fact that $T$ can be represented as a subspace of a product $\Pi B_\lambda$ of Banach spaces [12, pp. 46–47]. The weak topology of $\Pi B_\lambda$ is the same as the product topology when each $B_\lambda$ is given its weak topology. Since $T$ is complete, $T$ is strongly closed in $\Pi B_\lambda$. Since $T$ also is convex, $T$ is w-closed in $\Pi B_\lambda$. The canonical projection $E_\lambda$ of $E$ into $B_\lambda$ is bounded in $B_\lambda$. If $\operatorname{wcl}(E_\lambda)$, the weak closure of $E_\lambda$ as a subset of $B_\lambda$, were w-compact for each $\lambda$, it would follow from the Tychonoff theorem that the product $\Pi[\operatorname{wcl}(E_\lambda)]$ is compact in the product topology when each $B_\lambda$ is given its weak topology. This would imply that $E$ is w-compact, since $E$ being w-closed in this product follows from $E$ being w-closed in $T$ and $T$ being w-closed in $\Pi B_\lambda$. Thus at least one $\operatorname{wcl}(E_\lambda)$ is not w-compact in the corresponding $B_\lambda$. We can identify this $B_\lambda$ with a subset of a space $m(A)$ of bounded functions by letting $x$ be $\{f_\alpha(x)\}$, where for a suitable index set $A$ the set $\{f_\alpha : \alpha \in A\}$ is the set of all linear functionals on $B_\lambda$ with $\|f_\alpha\| \leqq 1$. Since $E_\lambda$ is bounded, $\operatorname{wcl}(E_\lambda)$ is bounded and is contained in a subset of $m(A)$ that is a product of compact intervals and therefore is compact in the product topology. On $B_\lambda$, the product topology of $m(A)$ is the weak topology of $B_\lambda$. Thus $\operatorname{wcl}(E_\lambda)$ is not closed in $m(A)$ with the product topology, since $\operatorname{wcl}(E_\lambda)$ is not w-compact. Let $w$ be a member of $m(A)$ that does not belong to $\operatorname{wcl}(E_\lambda)$, but belongs to the closure of $\operatorname{wcl}(E_\lambda)$ using the product topology. Then $w$ also belongs to the closure of $E_\lambda$ using the product topology. But $w$ is not in $B_\lambda$, since $\operatorname{wcl}(E_\lambda)$ is w-closed in $B_\lambda$. Let

$$\Delta = d(w, B_\lambda)$$

in the norm (sup) topology for $m(A)$, and choose $\theta$ with $0 < \theta < \Delta$. We shall show that it is possible to choose a sequence $\{(x_n, f_{\alpha_n})\}$ inductively so that each $f_{\alpha_n}$ is one of the members of $\{f_\alpha : \alpha \in A\}$ and, with the $\alpha$ component of $w$ in $m(A)$ denoted by $w_\alpha$:

(i) $x_n \in E_\lambda$, (iii) $f_{\alpha_n}(x_k) > \theta$ if $n \leqq k$,

(ii) $f_{\alpha_n}(x_k) = 0$ if $n > k$, (iv) $w_{\alpha_n} > \theta$.

Since $\|w\| > \theta$ and $w$ is in the product-closure of $E_\lambda$, it follows that $w_\alpha$ is a component of $w$ iff $-w_\alpha$ is a component, that there is a $w_{\alpha_1}$ with $w_{\alpha_1} > \theta$, and there is an $x_1$ in $E_\lambda$ with $f_{\alpha_1}(x_1) > \theta$. Now suppose that all $(x_k, f_{\alpha_k})$ have been chosen for $k < p$ in such a way that (i)–(iv) are satisfied for $k$ and $n$ less than $p$. We shall choose $\alpha_p$ so that $w_{\alpha_p} > \theta$ and $f_{\alpha_p}(x_k) = 0$ if $k < p$. To do this, we note first that we can define continuous linear functionals $x_k^c$ $(k < p)$ and $W$ on $B_\lambda^*$ by letting

$$x_k^c(f) = f(x_k) \text{ if } f \in B_\lambda^*, \qquad W(kf_\alpha) = kw_\alpha \text{ if } \alpha \in A.$$

Then it follows from

$$\left\| \sum_1^{p-1} a_i x_k^c + W \right\| = \sup \left\{ \left| \sum_1^{p-1} a_i x_k^c(f_\alpha) + W(f_\alpha) \right| : \alpha \in A \right\}$$

$$= \sup \left\{ \left| f_\alpha \left( \sum_1^{p-1} a_i x_k \right) + w_\alpha \right| : \alpha \in A \right\} = \left\| \sum_1^{p-1} a_i x_k + w \right\| \geqq \Delta,$$

that for any number $\Phi$ with $\theta < \Phi < \Delta$ the Helly's condition

$$\Phi \leqq \frac{\Phi}{\Delta} \left\| \sum_1^{p-1} a_i x_k^c + W \right\|$$

is satisfied for all numbers $\{a_i\}$, and therefore there is an $f$ in $B_\lambda^*$ for which

$$\|f\| \leqq 1, \quad W(f) = \Phi, \quad x_k^c(f) = 0 \text{ if } k < p.$$

Then $f = f_{\alpha_n}$ for some $\alpha_p$ in $A$ and $W(f) = w_{\alpha_p} = \Phi > \theta$ and $f_{\alpha_n}(x_k) = 0$ for $k < p$. We now have $w_{\alpha_n} > \theta$ if $n \leqq p$ and it follows from $w$ being in the product-closure of $E_\lambda$ in $m(A)$ that there is an $x_p$ in $m(A)$ with $f_{\alpha_n}(x_p) > \theta$ if $n \leqq p$. Then (i)–(iv) are satisfied for $k \leqq p$ and $n \leqq p$. Now that we have the sequence $\{(x_n, f_{\alpha_n})\}$, let $g_n$ be defined for each $n$ by $g_n(x) = f_{\alpha_n}(x_\lambda)$, where $x_\lambda$ is the component of $x$ in $B_\lambda$. Also, for each $x_n$ choose $z_n$ in $E_\lambda$ for which $x_n$ is the component of $z_n$ in $B_\lambda$. Then $g_n(z_k) > \theta$ if $n \leqq k$, and $g_n(z_k) = 0$ if $n > k$. The sequence $\{g_n\}$ is equicontinuous, since for any $\varepsilon > 0$ we have $|g_n(x)| < \varepsilon$ if the component of $x$ in $B_\lambda$ has norm less than $\varepsilon$.

There are other sequences of implications among (1)–(9) of Theorem 1 that can be proved easily. In particular, it would be easy to shorten the sequences used and show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (9) $\Rightarrow$ (1). Also, Theorem 2 could have been combined with Theorem 1. This was not done because it seemed best not to interrupt the chain of implications used in Theorem 1 and because Theorem 1 is long enough as it is. The equivalence of (1) and (10) is proved in [18] for $E$ bounded, closed and convex. Condition (12) is known [11, Theorem 6, p. 139]. Condition (14) was studied first for Banach spaces [14, Theorem 24, p. 581], but its equivalence with (1) for locally convex linear topological spaces is now well known [12, Theorem 17.12, p. 159]. Condition (16) is an interesting variation of the definition of weak sequential completeness in that $\lim g(x_n)$ is required to exist only for one $g$. Condition (17) is known for $T$ a Banach space and $K$ an arbitrary w-closed subset of $T$ (see [10, Theorem 1]).

THEOREM 2. *Let $T$ be a complete locally convex linear topological space and le $E$ be a bounded* w-*closed subset of $T$. Then the following are equivalent and each is equivalent to each of* (1)–(9) *of Theorem* 1.

(10) *Each* w-*continuous functional on E is bounded.*

(11) *Each bounded* w-*continuous functional on E attains its supremum on E.*

(12) *Each continuous linear functional attains its supremum on E.*

(13) *The closure of* cir$(E)$ *is* w-*compact.*

(14) *The closure of* conv$(E)$ *is* w-*compact.*

(15) *If* $\{x_n\}$ *is a sequence in E and* $\lim g(x_n)$ *exists for a particular* w-*continuous functional g, then there is an x in E with* $\lim g(x_n) = g(x)$.

(16) *If* $\{x_n\}$ *is a sequence in E and* $\lim g(x_n)$ *exists for a particular continuous linear functional g, then there is an x in E with* $\lim g(x_n) = g(x)$.

(17) *If K is a closed convex subset of T and E and K are disjoint, then* 0 *is not a member of* cl$[E - K]$.

**Proof.** If $\pi$ is an unbounded w-continuous functional on $E$, then the set of inverse images of the open intervals $(-n, n)$ is a w-open cover of $E$ that cannot be reduced to a finite cover. Thus (1) of Theorem 1 implies (10). The implication (10) $\Rightarrow$ (11) is trivial, since if $\pi$ is bounded and w-continuous and does not attain its supremum on $E$, then

$$\pi^*(x) = \frac{1}{\sup\{\pi(t) \colon t \in E\} - \pi(x)}$$

defines a w-continuous functional $\pi^*$ that is not bounded on $E$. The implication (11) $\Rightarrow$ (12) is formal, since a continuous linear functional is w-continuous. If each continuous linear functional attains its supremum on $E$, then this also is true for cir$(E)$. This follows from the fact that the supremum of $f$ on cir$(E)$ is equal to the supremum of $|f|$ on $E$, which follows from

$$f\left(\sum a_i x_i\right) \leqq \sup |f(x_i)| \quad \text{if} \quad \sum |a_i| \leqq 1.$$

Thus to prove (12) $\Rightarrow$ (13), we could show that cir$(E)$ is w-compact if each continuous linear functional attains its supremum on cir$(E)$. The proof of this is difficult and known and will not be given here (see [11, Theorem 6, p. 139]). The implications (13) $\Rightarrow$ (14) $\Rightarrow$ (1) follow from the fact that a closed subset of a compact set is compact.

Now the proof of (10) through (14) is complete. To show that (1) $\Rightarrow$ (15), we assume $E$ is compact and choose an arbitrary sequence $\{x_n\}$ in $E$ and suppose that $g$ is a w-continuous functional for which $\lim g(x_n)$ exists. Let this limit be $L$, and for each positive integer $n$ let

$$U_n = \left\{x \colon |L - g(x)| > \frac{1}{n}\right\}.$$

Then there is an $x$ in $E$ with $g(x) = L$, since otherwise the collection of all such sets $U_n$ would be a w-open cover of $E$ that cannot be reduced to a finite cover. The implication (15) $\Rightarrow$ (16) is purely formal and it is clear that (16) $\Rightarrow$ (12). Thus (15) and (16) are proved.

We shall show now that (1) $\Rightarrow$ (17). For each continuous linear functional $f$ and number $\Phi$ with $\sup\{f(y)\colon y \in K\} < \Phi$, let $W_f^\Phi$ be the w-open set $\{x\colon f(x) > \Phi\}$. For any $x$ in $E$, there is a continuous linear functional $f$ with $\sup\{f(y)\colon y \in K\} < f(x)$. Therefore the set of all $W_f^\Phi$ is a w-open cover of $E$. If $E$ is w-compact, this can be reduced to a finite cover. Choose $\varepsilon > 0$ so that

$$\Phi - \sup\{f(y)\colon y \in K\} > \varepsilon$$

for each $(f, \Phi)$ used in defining the finite cover and let $U$ be a neighborhood of 0 such that for each such $f$ we have $|f(x)| < \varepsilon$ if $x \in U$. Then it is impossible to have $x - y \in U$ with $x$ in $E$ and $y$ in $K$, since we would then have $x \in W_f^\Phi$ for some $(f, \Phi)$ and $|f(x-y)| < \varepsilon$, but also $f(x) > \Phi > f(y) + \varepsilon$. Thus 0 is not a member of $\mathrm{cl}[E - K]$ and (1) $\Rightarrow$ (17). To complete the proof of (17), we show that (17) $\Rightarrow$ (12). This is easy, since if $f$ is a continuous linear functional that does not attain its sup on $E$ and this sup is $m$, then we can let $K$ be $\{y\colon f(y) = m\}$ and it is easy to show that 0 is a member of $\mathrm{cl}[E - K]$.

2. **Convex bounded closed subsets of complete locally convex linear topological spaces.** Since a convex set is closed iff it is w-closed [12, Theorem 17.1, p. 154] we require that $E$ be closed and convex rather than w-closed and convex. Condition (19) of Theorem 3 is a generalization of a criterion given by Pták [19] for reflexivity of a Banach space, namely, that a Banach space is reflexive iff for each biorthogonal bounded sequence $\{(x_n, f_n)\}$ the sequence $\{x_1 + \cdots + x_n\}$ is unbounded. If (20) is modified by replacing $\bigcap H_n$ by $E \cap (\bigcap H_n)$, then the equivalence of (1) and (20) becomes another theorem of Pták [18]. Clearly the modified (20) can be sandwiched between (4) and (20). Condition (22) with $E$ the unit ball has been used as a characterization for reflexivity of a Banach space [16, Theorem 2, p. 1250]. Condition (23) is a strengthened form of (9), valid for closed convex sets.

THEOREM 3. *Let $T$ be a complete locally convex linear topological space and let $E$ be a convex bounded closed subset of $T$. Then the following are equivalent and each is equivalent to each of* (1)–(17) *of Theorems* 1 *and* 2.

(18) *If $\{(x_n, f_n)\}$ is a biorthogonal sequence for which some subsequence of $\{f_n\}$ is equicontinuous, then there is at least one value of $n$ for which $x_1 + \cdots + x_n$ is not in $E$.*

(19) *If $\{(x_n, f_n)\}$ is a biorthogonal sequence for which $\{x_n\}$ is bounded and $\{f_n\}$ is equicontinuous, then there is at least one value of $n$ for which $x_1 + \cdots + x_n$ is not in $E$.*

(20) *If $\{H_n\}$ is a sequence of closed hyperplanes and $E \cap H_1 \cap \cdots \cap H_n$ is nonempty for each $n$, then $\bigcap H_n$ is nonempty.*

(21) *For each sequence $\{x_n\}$ in $E$, the member* 0 *of $T$ belongs to*

$$\mathrm{cl}\left[\bigcup_{n=1}^{\infty}(\mathrm{lin}\{x_1,\cdots,x_n\}-\mathrm{flat}\{x_{n+1},\cdots\})\right].$$

(22) *Each affine continuous map of a nonempty, closed convex subset of E into itself has a fixed point.*

(23) *There does not exist a positive number $\theta$, a sequence $\{z_n\}$ in E, and an equicontinuous sequence $\{g_n\}$ of linear functionals such that*

$$g_n(z_k)=\theta \text{ if } n\leqq k, \qquad g_n(z_k)=0 \text{ if } n>k.$$

**Proof.** Clearly (18) $\Rightarrow$ (19). Let us prove that (5) $\Rightarrow$ (18). Suppose (18) is false and $\{(x_n,f_n)\}$ is as described in (18) with $\{f_{p_n}\}$ an equicontinuous subsequence of $\{f_n\}$, but that $x_1+\cdots+x_n$ is in $E$ for all $n$. Then (5) is false, since

$$\lim_n \lim_k f_{p_n}(x_1+\cdots+x_k)=1\neq 0=\lim_k \lim_n f_{p_n}(x_1+\cdots+x_k).$$

Now note that (20) is implied formally by (4) of Theorem 1, and (21) is implied by (8). Also, (22) is implied by (1). To see this, we use the fact that a continuous map of a convex compact subset of a locally convex linear topological space into itself has a fixed point [2, Theorem 1, p. 82]. Since $T$ is locally convex with the weak topology, to use (1) we need only know that a continuous affine map $\pi$ of a closed convex set $K$ into itself is w-continuous. This can be shown easily, since if $x$ is in $K$ and $f$ is a continuous linear functional, then the inverse image under $\pi$ of $\{t: f(t)\geqq f[\pi(x)]+\varepsilon\}$ is a closed convex subset of $K$ that can be separated strongly from $x$ by a hyperplane.

We shall show next that each of (19), (20), (21) and (22) implies (23), and that (23) implies (9) of Theorem 1. Suppose first that (23) is false and there is a positive number $\theta$, a sequence $\{z_n\}$ in $E$, and an equicontinuous sequence $\{g_n\}$ of linear functionals with $g_n(z_k)=\theta$ if $n\leqq k$ and $g_n(z_k)=0$ if $n>k$. Let $x_1=z_1$ and $x_n=z_n-z_{n-1}$ if $n>1$. Then the sequence $\{x_n\}$ is bounded and $x_1+\cdots+x_n$ equals $z_n$ and therefore belongs to $E$ for all $n$. Also, $g_n(x_n)=\theta$ and $g_n(x_k)=0$ if $n\neq k$, so the sequence $\{x_n, g_n/\theta\}$ is biorthogonal. Thus (19) is false. Now let $g_0(x)$ be defined as $\lim g_n(x)$ for all $x$ in $T$ for which this limit exists and then extended so as to be continuous on all of $T$ [12, Theorem 14.1 (iii), p. 118]. Let $H_0$ be the null set of $g_0$ and for each $n>0$ let $H_n$ be the set of all $x$ in $T$ with $g_n(x)=\theta$. Then $z_n$ belongs to $H_0\cap H_1\cap\cdots\cap H_n$ for all $n$, but if $x$ belongs to all $H_n$, then $g_n(x)=\theta$ for all $n>0$ and therefore $g_0(x)=\theta$ and $x\notin H_0$. Thus (20) is false. Now suppose that $W$ is a neighborhood of zero such that $|g_n(x)|<\theta$ for all $n$ if $x\in W$. Also suppose that $u-v$ is in $W$ and

$$u=\sum_1^n a_i z_i, \qquad v=\sum_{n+1} b_i z_i, \qquad \sum_{n+1} b_i=1.$$

Then $g_{n+1}(u-v)=g_{n+1}(-v)=-\theta$ and $u-v\notin W$, so we conclude that (21) is false. To contradict (22), we first show that if $x$ is in $\mathrm{cl}[\mathrm{conv}\{z_n\}]$ then $x = \sum_1^\infty \alpha_n z_n$ with each $\alpha_n$ nonnegative and $\sum \alpha_n = 1$. For each $x$ in $\mathrm{cl}[\mathrm{conv}\{z_n\}]$, we define $\alpha_n$ for each $n$ to be $[g_n(x) - g_{n+1}(x)]/\theta$. Since $g_1(w) = \theta$ if $w = \sum_1^p \beta_n z_n$ with $\sum \beta_n = 1$, and all of $|g_1(x) - g_1(w)|$ and $|\alpha_n - \beta_n|$ for $n > 0$ can be made small by a suitable choice of $w$ with each $\beta_n$ nonnegative, it follows that each $\alpha_n$ is nonnegative, $\sum_1^\infty \alpha_n \leqq 1$ and $\sum_1^\infty \alpha_i z_i$ is convergent, and $\theta = g_1(x) = \theta \cdot \sum_1^\infty \alpha_n$. Thus $\sum_1^\infty \alpha_n = 1$ and it follows that $x = \sum_1^\infty \alpha_n z_n$. Now let

$$\pi\left( \sum_1^\infty \alpha_n z_n \right) = \sum_1^\infty \alpha_n z_{n+1}.$$

Clearly $\pi$ has no fixed points. To show that $\pi$ is continuous on $\mathrm{cl}[\mathrm{conv}\{z_n\}]$, choose a particular sequence $\{\alpha_n\}$ with each $\alpha_n$ nonnegative and $\sum \alpha_n = 1$. For an arbitrary neighborhood $W$ of zero, choose a positive number $\delta$ and a circled neighborhood $W_0$ of zero such that

$$W_0 + W_0 \subset W, \qquad a \cdot \mathrm{cl}[\mathrm{conv}\{z_n\}] \subset W_0 \text{ if } |a| < \delta.$$

By approximating $\alpha_n$'s in a finite set whose sum is nearly one, we can see that there is a neighborhood $U$ of zero such that if $\sum \beta_n = 1$ and each $\beta_n$ is nonnegative, then $\sum |\alpha_n - \beta_n| < \delta$ if $\sum (\alpha_n - \beta_n) z_n \in U$. Then if $\sum (\alpha_n - \beta_n) z_n \in U$, we can write this sum as the sum of those terms with positive coefficients plus the sum of those terms with negative coefficients and obtain

$$\sum (\alpha_n - \beta_n) z_{n+1} \in W_0 + W_0 \subset W.$$

We must now show that (23) $\Rightarrow$ (9). To do this, we shall assume that $E$ is convex and (9) is false and then show that if $\{z_n\}$ is a sequence in $E$ and $\{g_n\}$ is an equicontinuous sequence of linear functionals with $g_n(z_k) > \theta$ if $n \leqq k$ and $g_n(z_k) = 0$ if $n > k$, then there is a sequence $\{u_n\}$ in $E$ and a sequence $\{h_n\}$ such that $h_n(u_k) = \frac{1}{2}\theta$ if $n \leqq k$, $h_n(u_k) = 0$ if $n > k$, and each $h_n$ is equal to $\Phi_n g_p$ for some $p$ and some positive number $\Phi_n < 1$. This can be done inductively as follows. Let $\underline{\lim}\, g_1(z_k) = \theta'$. If $g_1(z_k) = \theta'$ for all $k$, let $u_1 = z_1$ and $h_1 = (\theta/2\theta')g_1$. If $g_1(z_p) \neq \theta'$, choose a subsequence $\{z_{p_k}\}$ of $\{z_k\}$ with $p_k > p$ and $|\theta' - g_1(z_{p_k})|$ small enough for each $k$ that there is a number $\theta''$ near $\theta'$ and between $\theta'$ and $g_1(z_p)$ such that if $z_k^1$ is chosen for each $k$ so that

$$z_k^1 = \alpha z_p + (1-\alpha) z_{p_k} \text{ with } 0 \leqq \alpha < 1 \text{ and } g_1(z_k^1) = \theta'',$$

then $\alpha$ is small enough that

$$g_{p_n}(z_k^1) > \tfrac{3}{4}\theta \text{ if } n \leqq k.$$

Now the new sequence $\{z_k^1\}$ and the corresponding sequence $\{g_k^1\}$ of linear functionals from $\{g_n\}$ have all the original properties and in addition $g_1^1(z_k^1) = \theta''$ for all $k$. Now let $u_1 = z_k^1$ and $h_1 = (\theta/2\theta'')g_1^1$. We can then work with the new sequences $\{z_k^1: k > 1\}$ and $\{g_k^1: k > 1\}$ in exactly the same way to define $u_2$ and $h_2$, except that in the preceding inequality we replace 3/4 by 5/8. Continuing in this way, we get the desired sequences $\{u_n\}$ and $\{h_n\}$.

Several separation criteria are given in Theorems 4 and 5. These are closely related to (17) of Theorem 2. A condition similar to (25) is known to be a characterization of w-compactness for closed convex subsets of a Banach space (see [10, Theorem 2]). Condition (26) is closely related to a theorem that follows from results of Tukey [21] and Klee [13, p. 881]: *A Banach space is reflexive iff each pair of disjoint bounded closed convex subsets can be separated by a hyperplane.*

THEOREM 4. *Let T be a complete locally convex linear topological space and let E be a convex bounded closed subset of T. Then the following are equivalent and each is equivalent to each of* (1)–(23) *of the preceding theorems.*

(24) *If closed convex subsets X and Y of E are disjoint, then* 0 *is not a member of* $\mathrm{cl}[X - Y]$.

(25) *If closed convex subsets X and Y of E are disjoint, then there is a continuous linear functional f such that*

$$\sup\{f(x): x \in X\} < \inf\{f(y): y \in Y\}.$$

**Proof.** Since a closed convex set is w-closed, it follows from (1) that $X$ is w-compact. Then (24) follows from the equivalence of (1) and (17) when $E$ is replaced by $X$. We shall show that (24) $\Rightarrow$ (23). Suppose that (23) is false and therefore for some positive number $\theta$ there is a sequence $\{z_n\}$ in $E$ and an equicontinuous sequence $\{g_n\}$ of linear functionals such that $g_n(z_k) = \theta$ if $n \leqq k$ and $g_n(z_k) = 0$ if $n > k$. As for the set $\{\sum \alpha_n z_n\}$ discussed in the proof of the contradiction of (22) in the proof of Theorem 3, the following convex subsets of $E$ are closed:

$$X = \left\{ \sum_1^\infty \alpha_n \left( \frac{n}{n+2} \cdot z_{2n+1} + \frac{2}{n+2} \cdot z_{2n} \right) \right\},$$

$$Y = \left\{ \sum_1^\infty \alpha_n \left( \frac{n+1}{n+2} z_{2n-1} + \frac{1}{n+2} z_{2n} \right) \right\},$$

where each $\alpha_n$ is nonnegative and $\sum_1^\infty \alpha_n = 1$. Then $X \cap Y$ is empty, but 0 is a member of $\mathrm{cl}[X - Y]$.

The equivalence of (24) and (25) follows from the theorem on strong separation stated in the introduction.

THEOREM 5. *Let T be a complete locally convex linear topological space and let E be a convex bounded closed subset of T that contains* 0. *Then the following is equivalent to each of* (1)–(25) *of the preceding theorems.*

(26) *If closed convex subsets X and Y of E are disjoint, then there is a continuous linear functional f and a nonzero number* $\Phi$ *such that* $f(x) \leqq \Phi$ *if* $x \in X$ *and* $f(y) \geqq \Phi$ *if* $y \in Y$.

**Proof.** Clearly (25) $\Rightarrow$ (26). The proof that (26) implies (23) is similar to the part of the proof of Theorem 4 in which we showed that (24) $\Rightarrow$ (23), only now we let

$$X = \left\{ \sum_1^\infty \alpha_n \left( \frac{n}{n+2} z_{2n+1} + \frac{2}{n+2} z_{2n} \right) \right\},$$

$$Y = \left\{ \sum_1^\infty \alpha_n \left( \frac{n+1}{n+2} z_{2n-1} + \frac{1}{n+2} z_{2n} \right) \right\},$$

where each $\alpha_n$ is nonnegative and $\frac{1}{2} \leqq \sum_1^\infty \alpha_n \leqq 1$. Again $X$ and $Y$ are disjoint. Suppose there is a continuous linear functional $f$ and a nonzero number $\Phi$ such that $f(x) \leqq \Phi$ if $x \in X$ and $f(y) \geqq \Phi$ if $y \in Y$. Then for all $n$ we have

$$\frac{1}{2} f\left( \frac{n+1}{n+2} z_{2n-1} + \frac{1}{n+2} z_{2n} \right) \geqq \Phi, \quad f\left( \frac{n}{n+2} z_{2n-1} + \frac{2}{n+2} z_{2n} \right) \leqq \Phi.$$

This gives

$$f\left( \frac{n+1}{n+2} z_{2n-1} + \frac{1}{n+2} z_{2n} \right) + \left[ \Phi - f\left( \frac{n}{n+2} z_{2n-1} + \frac{2}{n+2} z_{2n} \right) \right] \geqq 2\Phi$$

and $f[(z_{2n-1} - z_{2n})/(n+2)] \geqq \Phi$ for all $n$, so that $\Phi \leqq 0$. Similarly,

$$\frac{1}{2} f\left( \frac{n}{n+2} z_{2n-1} + \frac{2}{n+2} z_{2n} \right) \leqq \Phi, \quad f\left( \frac{n+1}{n+2} z_{2n-1} + \frac{1}{n+2} z_{2n} \right) \geqq \Phi,$$

and $f[(z_{2n} - z_{2n-1})/(n+2)] \leqq \Phi$ for all $n$, so that $\Phi \geqq 0$. Since $\Phi \neq 0$, we conclude that (26) is false if (23) is false.

3. **Bounded w-closed subsets of Banach spaces.** The equivalence of condition (27) of Theorem 6 and (1) of Theorem 1 is the classic Eberlein theorem [3]. As will be clear from the proof, if we had wished only to obtain the Eberlein theorem we could easily have proved (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$(27) $\Rightarrow$ (9) $\Rightarrow$ (1). Condition (28) is known [10, Theorem 1, p. 204] and is included here largely because of its relation to (12) and to conditions for reflexivity given in Theorem 9.

THEOREM 6. *Let $E$ be a bounded* w-*closed subset of a Banach space. Then the following are equivalent and each is equivalent to each of* (1)–(17) *of Theorems* 1 *and* 2.

(27) *$E$ is* ws-*compact.*

(28) *If $S$ is a* w-*closed set and $E \cap S$ is empty, then $d(E,S) > 0$.*

**Proof.** First we assume (3) and let $\{x_n\}$ be an arbitrary sequence in $E$. Then there is an $x$ in $E$ such that

$$\underline{\lim} f(x_n) \leqq f(x) \leqq \overline{\lim} f(x_n)$$

for all continuous linear functionals $f$. Clearly $x$ is in $\mathrm{cl}[\mathrm{conv}\{x_n\}]$. Let $\{g_k\}$ be a sequence that is total over the closure of $\mathrm{conv}\{x_n\}$, and let $\{\xi_n\}$ be a subsequence of $\{x_n\}$ for which $\lim_n g_k(\xi_n)$ exists for each $k$. Then

$$\lim_n g_k(\xi_n) = g_k(x) \text{ for all } k.$$

Also, $x$ is a weak limit of $\{\xi_n\}$, since otherwise there would be a continuous linear functional $g$ for which $\lim g(\xi_n)$ does not exist or does not equal $g(x)$. Then we could choose a subsequence $\{\eta_n\}$ of $\{\xi\ \}$ for which $\lim g(\eta_n)$ exists and is not $g(x)$, and choose $y$ for which

$$\underline{\lim}\, f(\eta_n) \leqq f(y) \leqq \overline{\lim}\, f(\eta_n)$$

for all continuous linear functionals $f$. Then $y$ is in $\mathrm{cl}[\mathrm{conv}\{\eta_n\}]$. But also, $\lim_n g_k(\eta_n) = g_k(x)$ for all $k$, $\lim_n g_k(\eta_n) = g_k(y)$, and $g_k(x - y) = 0$ for all $i$. This is impossible, since $\{g_k\}$ is total over $\mathrm{cl}[\mathrm{conv}\{x_n\}]$ and $x \neq y$ follows from the two true statements: $\lim g(\eta_n) \neq g(x)$, and $\lim g(\eta_n) = g(y)$.

To show that (27)$\Rightarrow$(9), we let $\{z_n\}$ be an arbitrary sequence in $E$ and assume (27) so that some subsequence $\{\zeta_n\}$ of $\{z_n\}$ has a weak limit $w$. Then $w$ is in $\mathrm{cl}[\mathrm{conv}\{z_{n+1}, \cdots\}]$ for all $n$, since otherwise there would be an $n$ and a continuous linear functional $f$ with $\sup\{f(z_k)\colon k \geqq n+1\} < f(w)$ and thus $\lim f(\zeta_n) \neq f(w)$. Therefore for $\{z_n\}$ there can be no positive $\theta$ and bounded sequence $\{g_n\}$ of linear functionals such as described in (9), since then $g_n(z_k) > \theta$ for $n \leqq k$ would imply $g_n(w) \geqq \theta$ for all $n$, and $g_n(z_k) = 0$ if $n > k$ would imply $g_n(w) = 0$.

Condition (28) is an easy consequence of (27), since if $\lim \| x_n - y_n \| = 0$ with $x_n$ in $E$ and $y_n$ in $S$, then a weak limit of a subsequence of $\{x_n\}$ must belong both to $E$ and $S$. Also, (28)$\Rightarrow$(12). For if $f$ is a continuous linear functional that does not attain its sup on $E$ and $m$ is the sup of $f$ on $E$, then the set of all $x$ with $f(x) = m$ is w-closed and at zero distance from $E$.

4. **Reflexivity of Banach spaces.** Theorem 7 gives many characterizations of reflexivity for Banach spaces, since each of (1)–(28) can be used as a characterization of reflexivity if $E$ is taken to be the unit ball. The classical theorem that a Banach space is reflexive iff its unit ball is ws-compact is part of Theorem 7. The first step toward this theorem is given in Banach's book [1, Theorem 13, p.189] in which it is shown that if $B$ is separable and the unit ball is ws-compact, then $B$ is reflexive. In [5], it was proved that the unit ball of $B$ is ws-compact if $B$ is reflexive. The theorem was completed much later by Eberlein [3].

The proof given here that (31)$\Rightarrow$(29) was suggested by M. M. Day. A similar argument was used by Pták to prove theorems analogous to (19) (see [19, p.321]). A weaker form of (31) is known for which $g_n(z_k) = \theta$ is replaced by $g_n(z_k) \geqq \theta$ (see [9, Theorem 1, p. 206]). Condition (31) can also be stated in the following form (see [9, Corollary 1, p. 208]): *It is false that for each number $\theta < 1$ it is possible* *to embed B in a space of bounded functions defined on a set A in such a way that A contains the positive integers and, for each positive integer n, there is a member $z_n$ of B with*

$$z_n = (\theta, \theta, \cdots, \theta, 0, 0, \cdots; \{t_a^n\}),$$

*where the first n components of $z_n$ all are $\theta$ and $|t_a^n| \leqq 1$ for all a in A.*

THEOREM 7. *For a Banach space B, the following are equivalent and each is equivalent to each of* (1)–(28) *of the preceding theorems with E the unit ball.*

(29) *B is reflexive.*

(30) *It is false that for some positive number $\sigma$ there is a bounded sequence $\{z_n\}$ such that*

$$d(\operatorname{conv}\{z_1, \cdots, z_n\}, \operatorname{conv}\{z_{n+1}, \cdots\}) > \sigma \text{ for all } n.$$

(31) *It is false that for each number $\theta < 1$ there are sequences $\{z_n\}$ and $\{g_n\}$ with $\|z_n\| \leqq 1$, $\|g_n\| \leqq 1$, and*

$$g_n(z_k) = \theta \text{ if } n \leqq k, \; g_n(z_k) = 0 \text{ if } n > k.$$

**Proof.** With $E$ the unit sphere of $B$, it is clear that (30) is equivalent to (7). Therefore it is sufficient to show that (29) $\Rightarrow$ (30) $\Rightarrow$ (31) $\Rightarrow$ (29).

To show that (29) $\Rightarrow$ (30), suppose that (30) is false and that $\{z_n\}$ is a bounded sequence with

$$d(\operatorname{conv}\{z_1, \cdots, z_n\}, \operatorname{conv}\{z_{n+1}, \cdots\}) > \sigma \text{ for all } n.$$

Then for any $x$ in $B$, there is a $p$ such that $d(x, \operatorname{conv}\{z_{p+1}, \cdots\})$ is positive and therefore there is an $f$ in $B^*$ with $\sup\{f(z_n): n > p\} < f(x)$. Let $F$ be the member of $B^{**}$ described in the introductory lemma, so that

$$\underline{\lim} f(z_n) \leqq F(f) \leqq \overline{\lim} f(z_n) \text{ if } f \in B^*.$$

Then $B$ is not reflexive, since if there is an $x$ with $F(f) = f(x)$ for all $f$ in $B^*$, then

$$\underline{\lim} f(z_n) \leqq f(x) \leqq \overline{\lim} f(z_n) \text{ if } f \in B^*,$$

and this contradicts the fact that there is an integer $p$ and an $f$ in $B^*$ with $\sup\{f(z_n): n > p\} < f(x)$.

Now suppose that (31) is false. Choose a positive number $\theta < 1$ and let $\{z_n\}$ and $\{g_n\}$ be as described in (31). Then (30) is contradicted by $\{z_n\}$ with $\sigma$ any positive number less than $\theta$, since

$$\left| g_{n+1}\left( \sum_1^n \alpha_i z_i - \sum_{n+1} \beta_i z_i \right) \right| = \theta \text{ if } \sum \beta_i = 1.$$

To prove that (31) $\Rightarrow$ (29), we suppose that $B$ is not reflexive and let $B^c$ denote the canonical image of $B$ in $B^{**}$. Also, for each $x$ in $B$ let $x^c$ denote the canonical image of $x$ in $B^{**}$. Let $\theta$ satisfy $0 < \theta < 1$ and $F$ be a member of $B^{**}$ for which $\|F\| < 1$ and $d(F, B^c) > \theta$. The proof will be complete if we show that it is possible to choose the sequence $\{(z_n, g_n)\}$ inductively so that

(a) $\|z_n\| \leqq 1$ and $\|g_n\| \leqq 1$,

(b) $F(g_n) = \theta$ for all $n$,

(c) $g_n(z_k) = 0$ if $n > k$,

(d) $g_n(z_k) = \theta$ if $n \leqq k$.

The first step is to note that since $\|F\| > \theta$, there exists $g_1$ with $\|g_1\| \leqq 1$ and $F(g_1) = \theta$. Then $\|g_1\| > \theta$ and there exists $z_1$ with $\|z_1\| \leqq 1$ and $g_1(z_1) = \theta$. Now suppose that $(z_k, g_k)$ has been chosen for $k < p$ so that (a)–(d) are satisfied for $n$ and $k$ less than $p$. Then we choose $g_p$ so that $\|g_p\| \leqq 1$, $F(g_p) = \theta$, and $z_k^c(g_p) = g_p(z_k) = 0$ for $k < p$. This is possible, since the following Helly's condition is satisfied:

$$\theta \leqq \frac{\theta}{d(F, B^c)} \left\| \sum_1^{p-1} a_i x_i^c + F \right\| \text{ for all numbers } \{a_i\},$$

where $\theta / d(F, B^c) < 1$. Now we must choose $z_p$ so that $\|z_p\| \leqq 1$ and $g_n(z_p) = \theta$ if $n \leqq p$. To show this is possible, we use the fact that $\|F\| < 1$ and the Helly's condition:

$$\left| \sum_1^p a_i \theta \right| = \left| \sum_1^p a_i F(g_i) \right| = \left| F\left( \sum_1^p a_i g_i \right) \right| \leqq \|F\| \left\| \sum_1^p a_i g_i \right\|.$$

The following theorem gives several criteria for reflexivity that are closely related and might be called "flatness criteria" for the unit ball. These are closely related to conditions (7), (8), (21) and (30).

Condition (32) has long been known. It is the same as Lemma 1 of [8]. It is closely related to a necessary condition for reflexivity given by Milman and Milman [16, Corollary, p. 1252] which can be stated in the following form: *If $B$ is nonreflexive, then for any $\sigma < 1$ and any $n$ there is a sequence $\{z_1, \cdots, z_n\}$ such that $\sigma < \|u\| \leqq 1$ if $u \in \operatorname{conv}\{z_k\}$ and, for all $k < n$,*

$$\sigma < d(\operatorname{conv}\{z_1, \cdots, z_k\}, \operatorname{conv}\{z_{k+1}, \cdots, z_n\}) \leqq 1.$$

To change the Milman-Milman condition and obtain the necessary and sufficient condition (32), one can replace the finite sequence $\{z_1, \cdots, z_n\}$ by an infinite sequence and replace the 1 in the last inequality by 2 (which is equivalent to discarding it altogether).

Condition (35) is almost equivalent to the theorem of Pełczyński that a Banach space is nonreflexive iff some nonreflexive subspace has a basis [17, Theorem 1, p. 372], since the inequality being satisfied for *some* positive number $\frac{1}{2}\sigma$ is a necessary and sufficient condition for $\{z_n\}$ to be a basis for its closed linear span (see [1, p.111], and [7]).

Condition (35) also is related closely to Corollary 2 of [23], since a sequence is a "basic sequence of type $l_+$" iff it satisfies the conditions in (35) with $\| u \| \leqq 1$ replaced by $\| u \| \leqq M$ for some positive number $M$.

THEOREM 8. *For a Banach space, the following are equivalent and each is equivalent to each of* (1)–(31) *of the preceding theorems with E the unit ball.*

(32) *It is false that for each number* $\sigma < 1$ *there is a sequence* $\{z_n\}$ *such that* $\sigma < \| u \| \leqq 1$ *if* $u \in \text{conv}\{z_n\}$ *and, for all n,*

$$d(\text{conv}\{z_1, \cdots, z_n\}, \text{conv}\{z_{n+1}, \cdots\}) > \sigma.$$

(33) *It is false that for each number* $\sigma < 1$ *there is a sequence* $\{z_n\}$ *such that* $\sigma < \| u \| \leqq 1$ *if* $u \in \text{conv}\{z_n\}$ *and, for all n,*

$$d(\text{lin}\{z_1, \cdots, z_n\}, \text{ flat}\{z_{n+1}, \cdots\}) > \sigma.$$

(34) *It is false that for each number* $\sigma < 1$ *there is a sequence* $\{z_n\}$ *such that* $\sigma < \| u \| \leqq 1$ *if* $u \in \text{conv}\{z_n\}$ *and, for all n,*

$$d(\text{flat}\{z_1, \cdots, z_n\}, \text{lin}\{z_{n+1}, \cdots,\}) > \tfrac{1}{2}\sigma.$$

(35) *It is false that for each number* $\sigma < 1$ *there is a sequence* $\{z_n\}$ *such that* $\sigma < \| u \| \leqq 1$ *if* $u \in \text{conv}\{z_n\}$ *and, for all* $n < p$ *and all numbers* $\{a_i\}$,

$$\left\| \sum_1^{n+p} a_i z_i \right\| \geqq \tfrac{1}{2}\sigma \left\| \sum_1^{n} a_i z_i \right\|.$$

**Proof.** Clearly (30) implies (32), (33), (34) and (35). Also, (32) $\Rightarrow$ (33). We shall complete the proof by showing that each of (33) and (34) implies (31) and that (35) implies (29).

Suppose first that (31) is false and for $\theta < 1$ let $\{z_n\}$ and $\{g_n\}$ be as described in (31). We contradict (33) by using the first of the equalities

$$g_{n+1}\left( \sum_1^n a_i z_i - \sum_{n+1} b_i z_i \right) = -\theta \cdot \sum_{n+1} b_i,$$

$$g_1 \left( \sum_1^n a_i z_i - \sum_{n+1} b_i z_i \right) = \theta \cdot \left( \sum_1^n a_i - \sum_{n+1} b_i \right),$$

with $\sum_{n+1} b_i = 1$ and $\theta > \sigma$. Then (34) is contradicted by letting $\sum a_i = 1$ and noting that it is impossible to have both $\theta \left| \sum_{n+1} b_i \right|$ and $\theta \left| 1 - \sum_{n+1} b_i \right|$ less than $\sigma$ if $\theta > \sigma$.

To prove that (35) $\Rightarrow$ (29), we suppose $B$ is not reflexive and denote by $B^c$ the canonical image of $B$ in $B^{**}$. For $\Delta$ between 0 and 1, let $F$ be a member of $B^{**}$ for which $\| F \| < 1$ and $d(F, B^c) > \Delta$. The proof will consist of showing inductively that there is a linear functional $\Phi$ with domain $B$ and sequences $\{z_n\}$ and $\{H_n\}$ such that $\| \Phi \| = 1$, $F(\Phi) = \Delta$, and:

(i) $\| z_n \| \leqq 1$ and $\Phi(z_n) = \Delta$ for all $n$,

(ii) $\{H_n\}$ is an increasing sequence of finite sets of linear functionals with domains $B$ and norms less than $2/\Delta$,

(iii) $F(h) = h(z_p) = 0$ if $h \in H_n$ and $n < p$,

(iv) If $z \in \text{lin}\{z_1, \cdots, z_n\}$, there is an $h$ in $H_n$ with $|h(z)| \geqq \Delta \| z \|$.

First choose $\Phi$ so that $\| \Phi \| = 1$ and $F(\Phi) = \Delta$. Now suppose that $\{z_1, \cdots, z_p\}$ and $\{H_1, \cdots, H_p\}$ have been chosen to satisfy (i)–(iv) when $n \leqq p$, where $p$ may be zero. Then $z_{p+1}$ must be chosen so that $\| z_{p+1} \| \leqq 1$, $\Phi(z_{p+1}) = \Delta$, and $h(z_{p+1}) = 0$ if $h \in H_p$ (if $p = 0$, then $H_p$ is to be the empty set). This is possible, since $\| F \| < 1$ and the Helly's condition $\Delta \leqq \| F \| \, \| h + \Phi \|$ follows from

$$\Delta \leqq \frac{\Delta \| F \| \, \| h + \Phi \|}{|F(h + \Phi)|} = \| F \| \, \| h + \Phi \|, \text{ if } h \in H_p.$$

Now let $G_p$ be a finite set of linear functionals with unit norms and domains $\text{lin}\{z_1, \cdots, z_p\}$ which contains suitable linear functionals that for each $z$ in $\text{lin}\{z_1, \cdots, z_p\}$ there is a $g$ in $G_p$ with $|g(z)| \geqq \Delta \| z \|$. If $g \in G_p$ and $z_i^c$ is the canonical image of $z_i$ in $B^{**}$, then for all numbers $\{a_i\}$ we have

$$\left| \sum_1^p a_i z_i^c(g) \right| = \left| \sum_1^p a_i g(z_i) \right| = \left| g\left( \sum_1^p a_i z_i \right) \right| \leqq \left\| \sum_1^p a_i z_i \right\| = \left\| \sum_1^p a_i z_i^c \right\|$$

$$\leqq \left\| F + \sum_1^p a_i z_i^c \right\| + \left\| F \right\|.$$

Since $d(F, B^c) > \Delta$ and $\| F \| < 1$, we have $\Delta \| F \| < \| F + \sum_1^p a_i z_i^c \|$ and therefore

$$\left| \sum_1^p a_i z_i^c(g) \right| < \left( 1 + \frac{1}{\Delta} \right) \left\| F + \sum_1^p a_i z_i^c \right\|.$$

This is a Helly's condition that gives the existence of an $h$ in $B^*$ with $\| h \| < 2/\Delta$, $F(h) = 0$, and $z_i^c(h) = z_i^c(g)$, or $h(z_i) = g(z_i)$, for $1 \leqq i \leqq p$. Then $h$ is an extension of $g$ to $B$. Let each member of $G_p$ be extended in this way and then let $H_p$ be the union of $H_{p-1}$ and all such extensions of members of $G_p$. Now that we have sequences $\{z_n\}$ and $\{H_n\}$ that satisfy (i)–(iv), it follows from (i) that $\Phi(u) = \Delta$ and $\| u \| \geqq \Delta$ if $u \in \text{conv}\{z_n\}$, and it follows from (iv) that for any sum $\sum_1^n a_i z_i$ there is an $h$ in $H_n$ such that

$$\left| h\left( \sum_1^n a_i z_i \right) \right| \geqq \Delta \left\| \sum_1^n a_i z_i \right\|.$$

Using $\| h \| < 2/\Delta$ and (iii), we then have

$$\left\| \sum_1^{n+p} a_i z_i \right\| \geqq \frac{\Delta}{2} \left| h\left( \sum_1^{n+p} a_i z_i \right) \right| = \frac{\Delta}{2} \left| h\left( \sum_1^n a_i z_i \right) \right| \geqq \frac{\Delta^2}{2} \left\| \sum_1^n a_i z_i \right\|.$$

This contradicts (35), since for any $\sigma < 1$ we can choose $\Delta$ so that $\sigma < \Delta^2$.

All of the conditions in Theorem 9 are known; for (36) and (37) see [10, Theorems 3 and 4]; (38) is a consequence of results of Klee [13, p. 881] and Tukey [21].

Condition (36) is closely related to (17), (24) and (28). Conditions (37) and (38) are closely related to (25) and (26).

THEOREM 9. *For a Banach space, the following are equivalent and each is equivalent to each of* (1)–(35) *of the preceding theorems with E the unit ball.*

(36) *If* w*-closed subsets X and Y are disjoint and one set is bounded, then* $d(X, Y) > 0$.

(37) *If closed convex subsets X and Y are disjoint and one set is bounded, then there is a continuous linear functional f such that*

$$\sup\{f(x): x \in X\} < \inf\{f(y): y \in Y\}.$$

(38) *If closed bounded convex subsets X and Y are disjoint, then there is a continuous linear functional f and a nonzero number* $\Phi$ *such that* $f(x) \leqq \Phi$ *if* $x \in X$ *and* $f(y) \geqq \Phi$ *if* $y \in Y$.

**Proof.** Assuming (1) is true with $E$ the unit ball, the bounded set in (36) is w-compact and it then follows from (28) that $d(X, Y) > 0$. If $X$ and $Y$ also are convex and $d(X, Y) > \varepsilon$, then it follows from the theorem on strong separation stated in the introduction that (36) $\Rightarrow$ (37). Clearly (37) $\Rightarrow$ (38) and (38) is equivalent to (26) if $E$ is the unit ball.

## REFERENCES


1. S. Banach, *Thèorie des opérations linéaires*, Warsaw, 1932.

2. M. M. Day, *Normed linear spaces*, Academic Press, New York, 1962.

3. W. F. Eberlein, *Weak compactness in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 51–53.

4. M. Eidelheit, *Zur Theorie der konvexen Mengen in linearten normierten Räumen*, Studia Math. **6** (1963), 104–111.

5. V. Gantmaher and V. Smul'yan, *Sur les espaces linéaires dont la sphère unitaire est faiblement compacte*, Dokl. Akad. Nauk SSSR **17** (1937), 91–94.

6. A. Grothendieck, *Critères de compacticité dans les espaces fonctionnels généraux*, Amer. J. Math. **74** (1952), 168–186.

7. M. M. Grunblum, *Certain théorèmes sur la base dans un espace de type (B)*, Dokl. Akad. Nauk SSSR, **31** (1941), 428–432.

8. R. C. James, *Reflexivity and the supremum of linear functionals*, Ann. of Math. **66** (1957), 159–169.

9. R. C. James, *Characterizations of reflexivity*, Studia Math. **23** (1964), 205–216.

10. R. C. James, *Weak compactness and separation*, Canad. J. Math. **16** (1964), 204–206.

11. R. C. James, *Weakly compact sets*, Trans. Amer. Math. Soc. **113** (1964), 129–140.

12. J. L. Kelley and I. Namioka, *Linear topological spaces*, Van Nostrand, Princeton, 1963.

13. V. L. Klee, Jr., *Convex sets in linear spaces*, Duke Math. J. **18** (1951), 875–883.

14. M. Krein and V. Smul'yan, *On regularly convex sets in the space conjugate to a Banach space*, Ann. of Math. **41** (1940), 556–583.

15. D. Milman, *On some criteria for the regularity of spaces of the type (B)*, Dokl. Akad, Nauk SSSR **20** (1938), 243–246.

16. D. P. Milman and V. D. Milman, *Some geometric properties of nonreflexive spaces*, Soviet Math. **4** (1963), 1250–1252.

17. A. Pełczyński, *A note on the paper of I. Singer "Basic sequences and reflexivity of Banach spaces,"* Studia Math. **21** (1962), 371–374.

18. V. Pták, *Two remarks on weak compactness*, Czechoslovak Math. J. **5** (1955), 532–545.

19. V. Pták, *Biorthogonal systems and reflexivity of Banach spaces*, Czechoslovak Math. J. **9** (1959), 319–326.

20. V. Smul'yan, *On the principle of inclusion in the space of type (B)*, Mat. Sb. **5** (1939), 317–328.

21. J. W. Tukey, *Some notes on the separation of convex sets*, Portugal. Math. **3** (1942), 95–102.

22. K. Pełczyński, *A proof of Eberlein-Šmulian theorem by an application of basic sequences*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 539–544.

23. I. Singer, *Basic sequences and reflexivity of Banach spaces*, Studia Math. **21** (1962), 351–369.

Harvey Mudd College,
Claremont, California